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INSTABILITY OF STEADY FLOWS WITH CONSTANT VORTICITY IN VESSELS OF ELLIPTIC CROSS-SECTION*

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The problem of the stability of steady flows of a perfect incompressible fluid in vessels of elliptic cross-section is studied. The flow velocity field of the main stream is a linear function of the coordinates and the vorticity is constant. The spectral problem for the linear perturbations is solved using the mehtod of consecutive approximations. The instability of the flows to a first approximation is demonstrated. A special case of the flow in a triaxial ellipsoid is analysed in detail. Theoretical predictions agree well with the experimental results /1/. The present paper, unlike the analysis carried out in /1/, deals with an appreciably wider class of perturbations and the Galerkin method of rough a priori approximation is not used.

The problem of stability of flows of this type is of interest when describing the properties of a liquid-filled gyroscope /2—4/ and the behaviour of the star and planetary cores /5/. At the same time, a flow with a linear velocity field represents the simplest example of the realization of the new mechanism of instability of rotational flows connected with disturbance of the rotational symmetry /6—9/.

1. Formulation of the problem. A perfect incompressible fluid of uniform density completely fills a vessel, the boundary of which is described in a Cartesian system of coordinates x_0, y_0, z_0 by the relations

$$x_0^2/a^2 + y_0^2/b^2 + \varphi(z_0/c) = 0 \tag{1.1}$$

with three constants, a, b and c. Any intersection of the vessel by the plane $z_0 = \text{const}$ produces an ellipse with a ratio of the semi-axes equal to $a/b > 1$. The function $\varphi(z_0/c)$ is assumed to be non-positive and fairly smooth in some interval of variation of the argument z_0/c corresponding to the height of the vessel. For a vessel of finite height we can choose $-1 \leqslant z_0/c \leqslant 1$.

An exact solution of the equations of motion satisfying the conditions of impermeability at the surface (1.1) is given by the following expression for the velocity field $\mathbf{u}$:

$$\mathbf{u} = (-a\Omega y_0/b,\quad b\Omega x_0/a,\ 0) \tag{1.2}$$

Here Ω is a constant with dimensions of angular velocity. The flow (1.2) is characterized by constant vorticity which has a single non-zero component z_0. The vorticity establishes itself e.g. after a sudden stoppage of the vessel, filled with fluid and regarded as a rigid body.

The problem consists of investigating the stability of the flow (1.2) in the vessel (1.1) in the linear approximation.

The investigation is carried out using the "deformed" cylindrical coordinates r, θ, z, such, that

$$x_0/a = r\cos\theta,\quad y_0/b = r\sin\theta,\quad z_0/R = z$$
$$R \equiv ab/\sqrt{{}^1/_2(a^2+b^2)}$$

We also introduce the dimensionless time $\tau \equiv \Omega t$.

Since the basic flow (1.2) is stationary and the corresponding boundary conditions of impermeability at (1.1) hold, the problem of stability is reduced to a study of the perturbations depending harmonically on time. In particular, the pressure P perturbation field is given by

$$P(r, \theta, z, \tau) = p(r, \theta, z)\, e^{-i\omega\tau} \tag{1.3}$$

with complex amplitude function p and frequency ω.

Using well-known /1/ methods we obtain the following equation for p from the equations of motion:

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$$D^2\Delta p + 4p_{zz} = {}^1/_2\varepsilon\,(e^{2i\theta}N^+M^+ + e^{-2i\theta}N^-M^-)\,p \tag{1.4}$$

$$D \equiv -i\omega + \frac{\partial}{\partial\theta}, \quad \Delta \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}$$

$$N^{\pm} \equiv (D \pm 2i)(D \pm 4i), M^{\pm} \equiv M_1 \pm 2iM_2$$

$$M_1 \equiv \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

$$M_2 \equiv \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial r} - \frac{1}{r}\right), \quad \varepsilon \equiv \frac{a^2 - b^2}{a^2 + b^2}$$

Similarly we can show that at the boundary (1.1) described by the equation

$$r^2 + \varphi\,(z/h) = 0, \quad h \equiv c/R \tag{1.5}$$

the conditions of impermeability yield the relation

$$D\,(Dp_r + 2r^{-1}p_\theta) + [\varphi'/(rh)]\,(D^2 + 4)\,p_z = {}^1/_2\varepsilon\,(e^{2i\theta}N^+K^+ + e^{-2i\theta}N^-K^-)\,p \tag{1.6}$$

$$K^{\pm} \equiv \frac{\partial}{\partial r} \pm i\frac{\partial}{\partial\theta}, \quad \varphi'(\xi) \equiv \frac{d\varphi}{d\xi}$$

The problem (1.4)–(1.6) represents a spectral problem for determining the eigenfunctions $p(r, \theta, z)$ and eigenfrequencies ω. The existence of even a single eigenvalue ω with $\operatorname{Im}\omega > 0$ means that the flow (1.2) is unstable. Since the eigenvalues in (1.4)–(1.6) appear as complex conjugate pairs, the condition $\operatorname{Im}\omega \neq 0$, is sufficient for instability to occur.

Below we investigate problem (1.4)–(1.6) using the method of consecutive approximations, taking the smallness of ε into account. We assume that expansions of p and ω in integral powers of ε exist

$$(p, \omega) = \sum_{\nu=0}^{\infty} \varepsilon^\nu (p_\nu, \omega_\nu) \quad (\nu = 0, 1, 2, \ldots) \tag{1.7}$$

The problem of the convergence of the series (1.7) is not touched upon, and we confine our investigation to a detailed study of the first two approximations and to their comparison with experiment.

2. Zero approximation. The zero approximation solutions of problem (1.4)–(1.7) represent inertial waves in a liquid enclosed in an axisymmetric reservoir (1.5) and rotating as a rigid body. Since when $\varepsilon = 0$ the coefficients of the Eqs.(1.4), (1.6) are independent of θ, the problem reduces to investigating harmonics of the form

$$p_0\,(r, \theta, z) = A_m\,(r, z)\,e^{im\theta} \tag{2.1}$$

with integral wave numbers $m = 0, \pm1, \pm2, \ldots$. From (1.4)–(1.7) we obtain the following problem for A_m:

Problem. We have to find the solution of the problem

$$\frac{1}{r}(rA_{mr})_r - \frac{m^2}{r^2}A_m - \beta_m^2 A_{mzz} = 0 \tag{2.2}$$

$$\beta_m^2 \equiv (4 - \sigma_m^2)/\sigma_m^2, \quad \sigma_m \equiv m - \Omega_0$$

satisfying on (1.5) the boundary conditions

$$rA_{mr} + \frac{2m}{\sigma_m}A_m - \frac{\beta_m^2\varphi'}{2h}A_{mz} = 0 \tag{2.3}$$

Solution of the problem (2.2), (2.3) will enable us to determine the form of the eigenfunctions $A_m\,(r, z)$ and dispersion relations $\omega_0 = \omega_0\,(m, h)$.

In the general case when the vessel has an arbitrary shape, we know /10/ that the spectrum of the eigenvalues ω_0 of the problem (2.2), (2.3) is real and concentrated on the segment

$$m - 2 < \omega_0 < m + 2 \tag{2.4}$$

This implies, in particular, that $\beta_m^2 \geqslant 0$. We should also note the antisymmetry of the dispersion relation which follows from the form of (2.2), (2.3)

$$\omega_0\,(m, h) = -\omega_0\,(-m, h) \tag{2.5}$$

3. Instability in the general case. The instability to a first approximation ($\operatorname{Im}\omega_1 \neq 0$) can be illustrated for any function φ in (1.1). To do this, we must choose the zero approximation in the form of the sum of the harmonics (2.1) with $m = \pm1$:

$$p_0 = aA_1e^{i\theta} + \bar{a}A_{-1}e^{-i\theta} \tag{3.1}$$

Here a and $\bar{a}$ are independent complex constants. Expression (3.1) implies, by virtue of (1.3), (1.7), that the harmonics $m = 1$ and $m = -1$ have the same frequency ω_0. It is clear that such a degeneration does not occur for any value of the geometrical parameter h (1.5). The suitable values $h = h_0$ are found as the coordinates of the points of intersection of the families of dispersion curves $\omega_0(1, h)$ and $\omega_0(-1, h)$ in the plane of the variables ω_0, h. Such intersections occur by virtue of the overlap of the spectra for $m = 1$ and $m = -1$ on the segment $-1 < \omega_0 < 1$ (2.4). The antisymmetry of (2.5) corresponds to the intersections appearing on the axis $\omega_0 = 0$.

The basic result which will be obtained shortly, consists of the fact that for every point of intersection $\omega_0 = 0, h = h_0$ we have a corresponding range of values $|h - h_0| < \varepsilon h_*$, on which the flow (1.2) is unstable to a first approximation $(\mathrm{Im}\,\omega_1 \neq 0)$.

Substituting $\omega_0 = 0$ into (2.2), (2.3) we conclude that $A_{-1} = A_1 \equiv A$. After this we can rewrite (3.1) in the form

$$p_0 = (ae^{i\theta} + \bar{a}e^{-i\theta})A \tag{3.2}$$

where $A(r, z)$ satisfies the equation with boundary conditions at the surface (1.5)

$$L^+A \equiv \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} - 3\frac{\partial^2}{\partial z^2}\right)A = 0 \tag{3.3}$$

$$G^+A \equiv \left(r\frac{\partial}{\partial r} + 2 - \frac{3\varphi'}{2h}\frac{\partial}{\partial z}\right)A = 0 \tag{3.4}$$

The functional form of the solution to a first approximation follows from (3.2), and the appearance of the multipliers $e^{\pm 2i\theta}$ on the right-hand sides of (1.4), (1.6):

$$p_1 = Be^{i\theta} + \bar{B}e^{-i\theta} + Ce^{3i\theta} + \bar{C}e^{-3i\theta} \tag{3.5}$$

Here $B, \bar{B}, C, \bar{C}$ are four different functions of the arguments r and z, and are to be determined.

Eq.(1.4) to a first approximation (linear in ε) yields

$$\begin{aligned} L^+B &= a\omega_1 f_1 + \bar{a}f_2 \\ L^+\bar{B} &= -\bar{a}\omega_1 f_1 + af_2 \end{aligned} \tag{3.6}$$

$$f_1 \equiv 8A_{zz}, \quad f_2 \equiv \frac{9}{2}A_{zz}$$

Similarly, from the boundary conditions (1.6) we obtain

$$\begin{aligned} G^+B &= a(\omega_1 g_1 + h_1 g_3) + \bar{a}g_2 \\ G^+\bar{B} &= \bar{a}(-\omega_1 g_1 + h_1 g_3) + ag_2 \\ g_1 &\equiv 2\left(\frac{2\varphi_0'}{h_0}A_z - A\right) \\ g_2 &\equiv {}^3/_2(rA_r + A) \\ g_3 &\equiv -\frac{3}{2h_0^3}\left(\varphi_0' + \frac{z}{h_0}\varphi_0''\right)A_z \end{aligned} \tag{3.7}$$

In accordance with the results given above, we introduce into (3.7) $h = h_0 + \varepsilon h_1$, and the zero subscript accompanying the derivatives of the function φ means that the quantity h in its argument has been replaced by h_0.

The problem now consists of finding the solutions of inhomogeneous Eqs.(3.6) with inhomogeneous boundary conditions (3.7). We find, however, that we can determine ω_1 (and thus solve the most interesting part of the problem) without obtaining explicit expressions for B and $\bar{B}$. To do this we use the fact that the differential form $\Lambda \equiv rL^+$ is selfconjugate. The following Green's relation holds for any function and for v:

$$v\Lambda u - u\Lambda v = [r(vu_r - uv_r)]_r - 3r(vu_z - uv_z)_z$$

Selecting $v = A, u = B$, integrating the resulting equation over the area of meridianal cross-section S (1.5), transforming the integrals over the area S into line integrals over the boundaries ∂S and using the relations (3.3), (3.4), (3.6), (3.7), we can obtain the first of two equations

$$\begin{aligned} &a(\omega_1\varkappa_1 + h_1\varkappa_3) + \bar{a}\varkappa_2 = 0 \\ &\bar{a}(-\omega_1\varkappa_1 + h_1\varkappa_3) + a\varkappa_2 = 0 \\ &\varkappa_i \equiv F_i - V_i, \quad F_i \equiv \int_S Af_i r\,dr\,dz \\ &V_i \equiv \oint_{\partial S} Ag_i\,dz, \quad i = 1, 2, 3 \end{aligned} \tag{3.8}$$

The quantities f_1, f_2, g_1, g_2, g_3 are given by (3.6), (3.7); $f_3 \equiv 0$. The second equation of (3.8) is obtained in the same manner as the first.

The condition for a non-trivial solution $a, \bar{a}$ of system (3.8) to exist yields the expression

$$\omega_1^2 = (h_1^2\varkappa_3^2 - \varkappa_2^2)/\varkappa_1^2 \tag{3.9}$$

The minimum value of (3.9) is attained when $h_1 = 0$, and we always have the instability

$$\omega_1 = \pm i\omega_* \equiv \pm i\varkappa_2/\varkappa_1 \tag{3.10}$$

The interval of "unstable" values of h is given by the inequality

$$|h_1| < h_* \equiv |\varkappa_2/\varkappa_3| \tag{3.11}$$

4. Flow in a triaxial ellipsoid. We will extend the concepts given in Sect.3 by considering a problem of the stability of the flow (1.2) in a vessel in the shape of a triaxial ellipsoid. In this case we have in (1.1), (1.5)

$$\varphi = (z_0/c)^2 - 1 = (z/h)^2 - 1$$

The exact solution of (2.2), (2.3) for an ellipsoid can be written in terms of special functions /1/. With this in mind, we introduce the new variables ξ_m and η_m

$$\begin{aligned} r &= d_m[(1-\xi_m^2)(1-\eta_m^2)]^{1/2} \\ z &= \beta_m d_m \xi_m \eta_m, \quad d_m \equiv (1 + h^2/\beta_m^2)^{1/2} \end{aligned} \tag{4.1}$$

in which half of the ellipse $r^2 + z^2/h^2 \leqslant 1$, $r \geqslant 0$ is mapped onto a rectangle S_0:

$$-\alpha_m \leqslant \xi_m \leqslant \alpha_m, \quad \alpha_m \leqslant \eta_m \leqslant 1, \quad \alpha_m \equiv h/(h^2 + \beta_m^2)^{1/2} \tag{4.2}$$

In the new variables (2.2) will have a denumerable number of bounded solutions expressed in terms of the associated Legendre polynomials $P_l^{|m|}$:

$$A_{m,l} = P_l^{|m|}(\xi_m) P_l^{|m|}(\eta_m)$$

The index l can be equal to any integer greater than $|m| - 1$. The boundary condition (2.3) is reduced to the equation

$$\left(\frac{1-\alpha_m^2}{\alpha_m}\frac{d}{d\alpha_m} - \frac{2m}{\sigma_m}\right) P_l^{|m|}(\alpha_m) = 0 \tag{4.3}$$

from which we find α_m and the dispersion relation $\omega_0(m, h)$. Here we must take into account the relation

$$\sigma_m^2 = 4\alpha_m^2/[h^2 + (1 - h^2)\alpha_m^2]$$

Having written the solution in the zero approximation in the form of the sum of the harmonics

$$\sum_{m,l} a_{m,l} A_{m,l} e^{i(m\theta - \omega_0\tau)} \tag{4.4}$$

with some set of coefficients $a_{m,l}$, we encounter the problem of computing the first approximation. In order to solve the problem of stability completely, we must have the answer for any set of coefficients $a_{m,l}$. We find, however, that depending on what terms appear in (4.4), the quantity ω_1 may be real, as well as complex. We classify the cases using the following general assertions.

1°. If the zero approximation is chosen in the form of a single harmonic (a single term in (4.4)) with any value of m, then the quantity ω_1 is real.

2°. Let the sum (4.4) contain an arbitrary number of terms. If these terms do not contain terms with different m but the same ω_0 (no degeneration), then ω_1 are real.

3°. When we have such terms (the case of degeneracy), two different versions are possible. If these terms do not include two terms with $m = m_1$ and $m = m_2$, such that $|m_1 - m_2| = 2$, then ω_1 is real. If there are such terms, then ω_1 are always complex.

4°. The domains of instability ($\operatorname{Im}\omega_1 \neq 0$) in the plane h, ε have the form of "resonance" zones. The equation for each of these zones is written in the form $|h - h_0| \leqslant \varepsilon h_*$ with the constants h_0 and h_*. The quantity h_0 represents the value of h at which the degeneration mentioned above occurs with $|m_1 - m_2| = 2$.

The parts of the assertions formulated above referring to the existence of instability ($\operatorname{Im}\omega_1 \neq 0$), are proved by following, basically, the arguments of Sect.3. The assertions concerning the real character of ω_1 are simpler, and are obtained in the same manner.

Below we consider the simplest cases, to whose study the proof of instability can be reduced.

In accordance with what was said before, we write the function p_0 of the zero approximation for the "dangerous" perturbations in a form which generalized (3.1)

$$p_0 = aA(\xi, \eta) e^{i(n+1)\theta} + \bar{a}\bar{A}(\bar{\xi}, \bar{\eta}) e^{i(n-1)\theta} \tag{4.5}$$

where n is an arbitrary integer. In order to reduce the length of the expressions we have used in (4.5), and will continue to use from now on, the letters without indices and without a bar to denote the quantities with indices $n+1$, and letters with a bar to denote the quantities with indices $n-1$. Thus in (4.5) we have $a \equiv a_{n+1,l}$, $\bar a \equiv a_{n-1,\bar l}$

$$A \equiv A_{n+1,l}, \quad \bar A \equiv A_{n-1,\bar l}, \quad \xi \equiv \xi_{n+1}, \quad \eta \equiv \eta_{n+1}, \quad \bar\xi \equiv \xi_{n-1}$$

$\bar\eta \equiv \eta_{n-1}$; l and $\bar l$ are independent integers such that $l \geqslant |n+1|$, $\bar l \geqslant |n-1|$.

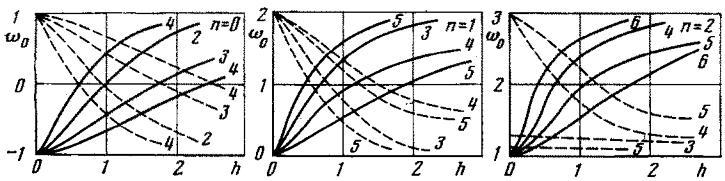

Fig.1

Expression (4.5) as well as (3.1), imply the existence of two solutions with the same frequency ω_0. This is possible not at any height h, but only for a denumerable set of values $h = h_0$, corresponding, in the plane ω_0, h, to the points of intersection of the dispersion curves $\omega_0(\bar n - 1, h)$ and $\omega_0(n+1, h)$. Fig.1 depicts the intersecting families of dispersion curves $\omega_0(m, h)$ (4.3) at $m = n \pm 1$ for $n = 0, 1, 2$. The solid lines represent the harmonic $n+1$, and the dashed lines the harmonic $n-1$. The number accompanying the curve denotes the value of l. Note that some pairs of values of m, l have two curves in Fig.1.

The functional form of the solution to a first approximation is constructed according to (4.5), (1.4), (1.6) and represents a generalization of (3.5)

$$p_1 = Be^{i(n+1)\theta} + \bar B e^{i(n-1)\theta} + Ce^{i(n+3)\theta} + \bar C e^{i(n-3)\theta}$$

The quantities $B, C; \bar B, \bar C$ denote the functions $\xi, \eta; \bar\xi, \bar\eta$ and are to be determined.

We write (3.6) in the following generalized form

$$\begin{aligned}
&LB = a\omega_1 f_1 + \bar a f_2 \qquad (4.6)\\
&\bar L \bar B = \bar a \omega_1 \bar f_1 + a \bar f_2\\
&L \equiv L_{n+1}(\xi, \eta), \quad \bar L \equiv L_{n-1}(\bar\xi, \bar\eta)\\
&L_m(\xi_m, \eta_m) \equiv N_m(\xi_m) - N_m(\eta_m)\\
&N_m(x) \equiv (1-x^2)\frac{\partial^2}{\partial x^2} - 2x\frac{\partial}{\partial x} - \frac{m^2}{1-x^2}\\
&f_1 \equiv \frac{8d^2}{\sigma^3}(\xi^2 - \eta^2) A_{zz}, \quad \bar f_1 \equiv \frac{8\bar d^2}{\bar\sigma^3}(\bar\xi^2 - \bar\eta^2)\bar A_{zz}\\
&f_2 \equiv d^2(\xi^2 - \eta^2)\frac{\sigma+2}{2\sigma}\left[\bar\beta^2 \bar A_{zz} + \frac{2n}{r}\left(\frac{n-1}{r} - \frac{\partial}{\partial r}\right)\bar A\right]\\
&\bar f_2 \equiv \bar d^2(\bar\xi^2 - \bar\eta^2)\frac{\bar\sigma - 2}{2\bar\sigma}\left[\beta^2 A_{zz} + \frac{2n}{r}\left(\frac{n+1}{r} + \frac{\partial}{\partial r}\right)A\right]\\
&\sigma \equiv \sigma_{n+1}, \quad \bar\sigma \equiv \sigma_{n-1}, \quad \beta \equiv \beta_{n+1}, \quad \bar\beta \equiv \beta_{n-1}\\
&d \equiv d_{n+1}, \quad \bar d \equiv d_{n-1}
\end{aligned}$$

where d_m, β_m, σ_m are taken from (4.1), (2.2).

Similarly, the boundary conditions (1.6) imply that the following relations hold at the boundary of the rectangle S_0 (4.2):

$$\begin{aligned}
&GB = \alpha^2(\xi^2 - \eta^2)[a(\omega_1 g_1 + h_1 g_3) + \bar a g_2] \qquad (4.7)\\
&\bar G \bar B = \bar\alpha^2(\bar\xi^2 - \bar\eta^2)[\bar a(\omega_1 \bar g_1 + h_1 \bar g_3) + a \bar g_2]\\
&G \equiv G_{n+1}(\xi, \eta), \quad \bar G \equiv G_{n-1}(\bar\xi, \bar\eta)\\
&G_m(\xi_m, \eta_m) \equiv (\alpha_m^2 - \xi_m^2) Q_m(\eta_m) - (\alpha_m^2 - \eta_m^2) Q_m(\xi_m)\\
&Q_m(x) \equiv (1-x^2)x\frac{\partial}{\partial x} - \frac{2m\alpha_m^2}{\sigma_m}\\
&g_1 \equiv \frac{2}{\sigma^2}\left[\frac{4}{\sigma h_0^2} zA_z - (n+1)A\right]\\
&\bar g_1 \equiv \frac{2}{\bar\sigma^2}\left[\frac{4}{\bar\sigma h_0^2} z\bar A_z - (n-1)\bar A\right]\\
&g_2 \equiv \frac{\sigma+2}{2\sigma}[r\bar A_r - (n-1)\bar A], \quad g_3 \equiv -\frac{2\beta^2}{h_0^3} zA_z\\
&\bar g_2 \equiv \frac{\bar\sigma - 2}{2\bar\sigma}[rA_r + (n+1)A], \quad \bar g_3 \equiv -\frac{2\bar\beta^2}{h_0^3} z\bar A_z
\end{aligned}$$

In order to make the expressions shorter, we have retained on the right-hand sides of (4.6), (4.7) the derivatives in r and z, which can be expressed in the form of simple but bulky expressions in terms of the derivatives in ξ_m and η_m. In addition, in accordance with the results formulated in assertion 4°, we have introduced $h = h_0 + \varepsilon h_1$ in (4.7), i.e. we assumed that h differs little from the "resonance" value h_0.

Table

n	l	ω_0	h_0	ω_*	h_*
0	2	0	1	0,5	0,5
0	3	0	1,803	0,518	1,634
0	4	0	2,529	0,524	3,341
0	4	0,548	1,315	0,015	0,060
0	4	0	0,593	0,547	0,709
1	3	1,087	0,772	0,519	0,643
1	4	1,062	1,382	0,453	1,205
1	5	0,876	1,571	0,455	10,416
1	5	1,389	0,793	0,460	3,763
1	5	0,550	1,037	0,251	1,159
1	5	1,012	0,486	0,632	0,847
2	4	2,110	0,645	0,521	0,577
2	5	2,078	1,157	0,967	3,154
2	5	1,069	0,190	0,761	2,101

Below we use Green's equation which can be written for a selfconjugate operator L, for example, in the form

$$vLu - uLv = [(1-\xi^2)(vu_\xi - uv_\xi)]_\xi - [(1-\eta^2)(vu_\eta - uv_\eta)]_\eta$$

Putting $v = A,\ u = B$, integrating the resulting equation over the area S_0 (4.2) and repeating the procedure used in deriving (3.8), we can obtain the following equations:

$$a(\omega_1\varkappa_1 + h_1\varkappa_3) + \bar{a}\varkappa_2 = 0 \tag{4.8}$$

$$\bar{a}(\omega_1\bar{\varkappa}_1 + h_1\bar{\varkappa}_3) + a\bar{\varkappa}_2 = 0$$

$$\varkappa_i \equiv F_i + V_i, \quad F_i \equiv \int_{S_0} f_i A\, d\xi\, d\eta, \quad f_3 \equiv 0,$$

$$V_i \equiv \alpha\Big(\int_{-\alpha}^{\alpha} g_i A\,|_{\eta=\alpha}\, d\xi + \int_{\alpha}^{1} g_i A\,|_{\xi=\alpha}\, d\eta + \int_{\alpha}^{1} g_i A\,|_{\xi=-\alpha}\, d\eta\Big), \quad i = 1, 2, 3$$

The expressions for $\bar{\varkappa}_i$, are obtained by adding a bar to $\alpha, A, \xi, \eta, f_i, g_i$ in the expressions for $\varkappa_i$.

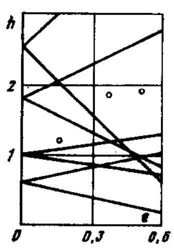

Fig.2

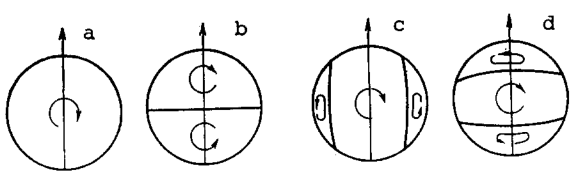

Fig.3

The condition for a non-trivial solution of (4.8) to exist leads to the expression

$$\omega_1 = \{-h_1(\varkappa_1\bar{\varkappa}_3 + \bar{\varkappa}_1\varkappa_3) \pm [h_1^2(\varkappa_1\bar{\varkappa}_3 - \bar{\varkappa}_1\varkappa_3)^2 + 4\varkappa_1\bar{\varkappa}_1\varkappa_2\bar{\varkappa}_2]^{1/2}\}/(2\varkappa_1\bar{\varkappa}_1)$$

The lowest value of the expression under the radical sign is reached when $h_1 = 0$. The instability corresponds to $\varkappa_2\bar{\varkappa}_2/\varkappa_1\bar{\varkappa}_1 < 0$. In this case we have, when $h_1 = 0$

$$\omega_1 = \pm i\omega_*, \quad \omega_* \equiv \sqrt{-\varkappa_2\bar{\varkappa}_2/\varkappa_1\bar{\varkappa}_1} \tag{4.9}$$

We have instability when

$$h_1 < h_* \equiv 2\sqrt{-\varkappa_1\bar{\varkappa}_1\varkappa_2\bar{\varkappa}_2}/|\varkappa_1\bar{\varkappa}_3 - \bar{\varkappa}_1\varkappa_3| \tag{4.10}$$

When $n = 0$, some of the points of intersection of the dispersion curves lie on the axis $\omega_0 = 0$ (Fig.1). We have here $\bar{\varkappa}_1 = -\varkappa_1,\ \bar{\varkappa}_2 = \varkappa_2,\ \bar{\varkappa}_3 = \varkappa_3$ and the instability follows automatically from (4.9), (4.10). This result represents a special case of (3.10), (3.11).

5. Results of computations. Comparison with experiment. The quantities ω_0, h_0, ω_*, h_* were computed for all points of intersection of the curves $\omega_0(n \pm 1, h)$ appearing in

Fig.1. The method of selecting the first few curves from their denumerable set is connected with the fact that the perturbations with smallest values of l (large scale perturbations) are best suited to illustrate and explain the experimental facts /1/.

After determining the values of ω_0, h_0, we computed ω_* (4.9) and h_* (4.10). We found that all points of intersection can be divided into two groups. The first group contains the points of intersection of the curves with different values of l, and we have $\omega_* = 0$ for all these points, which corresponds to neutral stability to a first approximation. The second group contains all points of intersection of the curves with the same value of l. Here ω_* are real, and the perturbation amplitudes to a first approximation always increase exponentially. The table gives the values of $\omega_0, h_0, \omega_*, h_*$ for points of intersection belonging to the second group.

The general conclusion emerging from a comparison of the quantities ω_* and h_* at various different points (see table) consists of the fact that there is no weakening of instability as n or l increase. At the same time, the experiments quoted in /1/ showed the instabilities only for $n = 0$ and $\omega_0 = 0$. This particular feature emerging when the theory is compared with experiment remains, so far, unexplained.

When $n = 0, \omega_0 = 0$, we have good agreement between the results predicted by the theory and those obtained experimentally. Fig.2 shows the zones of instability $|h - h_0| \leqslant \varepsilon h_*$ in the plane h, ε, corresponding to the cases $n = 0, \omega_0 = 0$ of the table. Fig.3 shows qualitative patterns indicating the directions of the velocity field of the unstable modes, in the meridianal cross-section of the ellipsoid for every zone shown in Fig.2. Fig.3a corresponds to $h_0 = 1, l = 2$; 3b corresponds to $h_0 = 1{,}803, l = 3$; $b - h_0 = 0{,}593, l = 4$; and 3c corresponds to $h_0 = 2{,}592, l = 4$. The cases $l = 2, 3, 4$ correspond, according to the terminology of /1/, to the single-, double- and triple-vortex instability. The correspondence between the patterns of Fig.3 and zones in Fig.2 is established simply by comparing the values of h_0. The points in Fig.2 denote the values of h, ε from the experiments in /1/. Two upper points appear almost in the middle of the zone corresponding to Fig.3b. In agreement with this theoretical prediction, the experiments show a two-vortex instability. The lower point in Fig.2 corresponds, in theory as well as in practice, to stable rotation of the liquid. Agreement between the theory and experiment is also shown in the correct prediction that the unstable vortical structure has zero velocity of rotation $\omega_0 = 0$.

Comparing the results of the proposed theory with conclusions drawn in /1/, we must stress three points. Firstly, the predictions made in /1/ concern only the cases $n = 0, \omega_0 = 0$. Secondly, in these cases the parameters h_0 are the same in both theories, but h_* are, generally speaking, different. Thirdly, the experimental data available at present agree with /1/, as well as with the theory given here.

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